PMM U.S.S.R.,Vol.45,pp.710-713
0021-8928/82/50710 $7.50/0

UDC 531.011:629.113 -629.735

METHOD OF DETERMINING SINGULAR POINTS AND THEIR PROPERTIES IN THE PROBLEM OF PLANE MOTION OF A WHEELED VEHICLE*

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A method is proposed for the determination of singular points in the problem of plane-parallel motion of a wheeled vehicle (automobile, aircraft on the runway, etc.) which is valid for any arbitrary dependence of lateral reactions on the slip angle. A combination of this method with the Poincaré theory of indices enables us to determine the nature of behavior of the zero solution of the equation of perturbed motion. It is shown that in the case of arbitrary law of lateral slip the loss of coordinate origin stability corresponds either to the creation of a multiple singular point or to the merging of singular points there. The singular case of cubic approximation of lateral reactions is considered.

According to /1/ the dynamic behavior of a model of a wheeled vehicle moving at constant speed in a straight line on a horizontal plane is defined by two variables: the transverse velocity of the center of mass and the angular yaw velocity. The driver cannot directly affect these quantities; the vehicle controllable parameters are the front wheel deviation angle (here assumed zero) and the wheel longitudinal velocity (assumed constant). In this formulation the problem corresponds to that in /2/.

Linearization of the dependence of lateral force on the slip angle, used in the majority of investigations of the dynamics of railess vehicles, has at least two shortcomings. On the one hand, it does not reflect the experimental data whose great majority explicitly show their nonlinearity, and on the other, the linear model makes it impossible to explain the existence of the vehicle critical velocity in the case of oversteer and its absence in that of vehicles with understeer.

The natural and simplest development of the linear theory is the cubic approximation of lateral reactions Y_i with respect to the slip angles δ_i, which was used in /3,4/ and is applicable when the curve $Y_i = Y_i(\delta_i)$ is convex. However, that curve has often inflection points and a fairly complex configuration. It is sometimes possible at the design stage of the vehicle to vary function $Y_i(\delta_i)$ by suitable selection of the vehicle parameters. These aspects make pressing the investigation of vehicle motion in the case of an arbitrary law of slip.

A geometric method of determination of singular points by the form of function $Y_i = Y_i(\delta_i)$, with the stability analysis carried out on the basis of the phase pattern obtained by the method of isoclines is proposed in /5/. The method imposes substantial constraints on the vehicle parameters.

We shall show that in investigation of rectilinear motion stability it is possible to avoid the construction of phase curves by combining the graphic method of /5/ with the theory of singular points indices which states that the loss of rectilinear motion stability is connected with the appearance of a multiple singular point at the coordinate origin.

If m is the mass, I is the moment of inertia of the vehicle about the vertical axis passing through its center of mass D, v, u are the longitudinal and transverse velocities of point D, l_1, l_2 are the distances of point D from the middle of the front and rear axes, respectively, and ω is the angular yaw velocity, then the equations of motion are of the form /1/

$$m(u^{\cdot} + v\omega) = Y_1 + Y_2, \quad I\omega^{\cdot} = Y_1 l_1 - Y_2 l_2 \tag{1}$$

The determination of singular points reduces to the solution of system

$$mv\omega = Y_1 + Y_2, \quad Y_1 l_1 = Y_2 l_2 \tag{2}$$

Let N_i be the vertical load on the i-th axis. Then $N_1 = mgl_2/l$, $N_2 = mgl_1/l$. Hence the

*Prikl.Matem.Mekhan.,45,No.5,944-948,1981 second of equations of system (2) is of the form $Y_1N_2 = Y_2N_1$. Introducing in the analysis the quantity Y such that

$$Y_1/N_1 = Y_2/N_2 = Y \tag{3}$$

we obtain from the first of Eqs. (2) $Y = v\omega/g$. Since $\omega = (\delta_2 - \delta_1)\,v/l$, hence

$$Y = v^2 g^{-1} l^{-1} (\delta_2 - \delta_1) \quad (l = l_1 + l_2) \tag{4}$$

The slope of curve (4) in the plane of Cartesian coordinates of Y, and $\delta_2 - \delta_1$ is always positive and increases to $+\infty$ as the motion velocity increases. It follows from (3) that Y is some known function of variables $\delta_2 - \delta_1$, i.e.

$$Y = Y(\delta_2 - \delta_1) \tag{5}$$

The intersection points of the stright line (4) with curve (5) correspond to singular points of system (1).

Let us illustrate the method on the example of linear hypothesis of slip $Y_1 = a_1\delta_1$, $Y_2 = a_2\delta_2$, where a_i are the coefficients (constant) of resistance to slip. In this case (5) is the straight line

$$Y = k_1k_2(k_1 - k_2)^{-1}(\delta_2 - \delta_1), \quad k_1 = a_1 l/(mgl_2), \quad k_2 = a_2 l/(mgl_1) \tag{6}$$

When $k_1 < k_2$, i.e. when $a_1l_1 - a_2l_2 < 0$, the straight line (6) can intersect the straight line (4) only at the coordinate origin. Thus for any v there is only one singular point at the

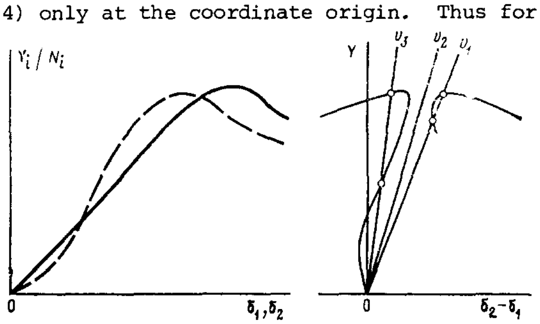

Fig.1

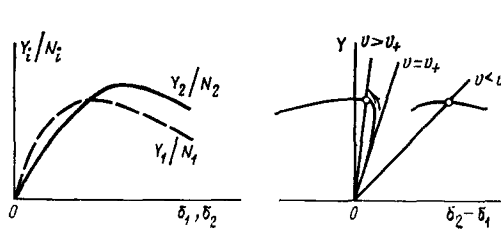

Fig.2

coordinate origin, which corresponds to stability of motion at all velocities. If, however, $k_1 > k_2$, i.e. $a_1l_1 - a_2l_2 > 0$, then the "movable" straight line (4) approaches the straight line (6) as v is increased. At certain velocity $v = v_+$ there is a whole straight line of singular points, which corresponds to the loss of stability of the zero solution of equations of perturbed motion.

Since usually the law of slip at small slip angles is linear, the Liapunov stability of the coordinate origin is determined in nonsingular cases by the linear part. In the case shown in Fig.1 (in which and other diagrams below the dash and solid lines correspond to $i = 1$ and $i = 2$, respectively) we have

$$k_2 = [d(Y_2/N_2)/d\delta_2]_{\delta_2=0} > k_1 = [d(Y_1/N_1)/d\delta_1]_{\delta_1=0} \tag{7}$$

But the analysis of the linear model implies that when $k_2 > k_1$ the rectilinear motion is stable at any v. Hence (7) predetermines the coordinate origin stability at any v.

Consequently, in the case of an arbitrary hypothesis about slip, either a singular point is generated at the coordinate origin or singular points merge there, i.e. the problem of stability reduces to the analysis of behavior of singular points. Let us illustrate this statement on some examples.

1°. **Generation of a multiple singular point.** Analysis of the right side of Fig. 2 shows that when $v < v_+$ there are no singular points in the small neighborhood of point (0,0), the coordinate origin is an isolated singular point and the Poincaré index of the coordinate origin is unity when $v < v_+$. Hence for the remaining values of parameter v the sum of indices of all singular points must also be unity. When $v = v_+$, a multiple singular point is generated at the coordinate origin. When $v > v_+$ there are two "mobile" (i.e. dependent of velocity v) singular points in the first and third quadrants of the coordinate origin neighborhood. Since the index of the coordinate origin is -1 (a saddle) when $v > v_+$, the index of every mobile point is equal 1.

Let us show that *in some neighborhood of the coordinate origin*, i.e. when $v_+ < v < v_+ + v'$, where $v' > 0$, these mobile singular points are stable. For this we substitute in the equations of motion (1) δ_1, δ_2 for u, ω using formulas

$$u = -vl^{-1}(l_2\delta_1 + l_1\delta_2) + \ldots, \quad \omega = vl^{-1}(\delta_2 - \delta_1) + \ldots$$

in new variables the equations of motion are of the form

$$\begin{aligned}
&\delta_1^{\cdot} = P(\delta_1, \delta_2),\ \delta_2^{\cdot} = Q(\delta_1, \delta_2)\\
&P = -[v/l + a_1v^{-1}(m^{-1} + l_1^2I^{-1})]\delta_1 + [v/l + a_2v^{-1}(l_1l_2I^{-1} - m^{-1})]\delta_2 + \ldots\\
&Q = [a_1v^{-1}(l_1l_2I^{-1} - m^{-1}) - vl^{-1}]\delta_1 + [v/l - a_2v^{-1}(l_2^2I^{-1} + m^{-1})]\delta_2 + \ldots\\
&a_1 = (dY_1/d\delta_1)_{\delta_1=0},\ a_2 = (dY_2/d\delta_2)_{\delta_2=0}
\end{aligned}$$

Hence

$$-p|_{(0,0)} = [\operatorname{div}(P, Q)]_{(0,0)} = \left(\frac{\partial P}{\partial \delta_1} + \frac{\partial Q}{\partial \delta_2}\right)_{(0,0)} = -\frac{1}{v}\left(\frac{a_1 + a_2}{m} + \frac{a_1l_1^2 + a_2l_2^2}{I}\right)$$

$$q|_{(0,0)} = \left[\frac{D(P, Q)}{D(\delta_1, \delta_2)}\right]_{0,0} = \begin{vmatrix} \partial P/\partial\delta_1 & \partial P/\partial\delta_2 \\ \partial Q/\partial\delta_1 & \partial Q/\partial\delta_2 \end{vmatrix}_{(0,0)} = \frac{a_1a_2l^2 - mv^2(a_1l_1 - a_2l_2)}{mv^2I}$$

Always $p > 0$. If $a_1l_1 - a_2l_2 \leqslant 0$, then $q > 0$ for any v. When $a_1l_1 - a_2l_2 > 0$ we have

$$q = (a_1l_1 - a_2l_2)I^{-1}v^{-2}(v_+^2 - v^2), \quad v_+ = l[a_1a_2m^{-1}(a_1l_1 - a_2l_2)^{-1}]^{1/2}$$

Hence $q > 0$ when $v < v_+$ and $q < 0$ when $v > v_+$. Owing to the continuity of function $\operatorname{div}(P, Q)$ *with respect to both variables* δ_1, δ_2 *in the determination region* $\exists\alpha > 0$: $|\delta_1^*| < \alpha$, $|\delta_2|^* < \alpha$ (i.e. $v_+ < v < v_+ + v'$, $v' > 0) \Rightarrow [\operatorname{div}(P, Q)]_{(\delta^*_1, \delta_2^*)} > 0$.

Since the Poincaré index of the singular point (δ_1^*, δ_2^*) is unity, these points are stable and are either nodes or focuses, Q.E.D.

Thus, in spite of the coordinate origin being an unstable singular point (a saddle), the perturbations do not exceed in this case a certain finite value. Behavior of phase curves is shown in Fig.3 in the case when the stationary singular points are focuses.

2°. **Merging of singular points.** This case is represented in Fig.4. When $v < v_+$ the coordinate origin index is 1, while for $v < v_+$ there are two moving singular points (in the first and third quadrant). When $v = v_+$ these mobile points merge at the coordinate origin. Since when $v > v_+$ the coordinate origin is a saddle, the sum of indices of all singular points as $v < v_+$ is -1. Hence the mobile singular points are of the saddle type. In this case, with $v > v_+$ the perturbation growth is unbounded.

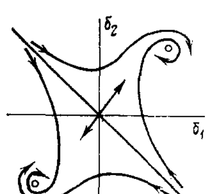

Fig.3

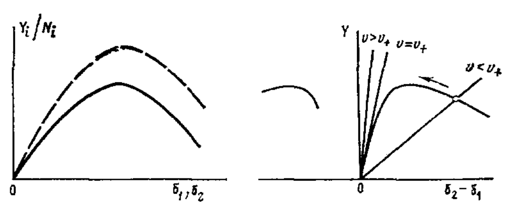

Fig.4

The particular case of cubic approximation of lateral reactions. Let us illustrate the method on the example of $Y_i = a_i\delta_i - b_i\delta_i^3$ $(i = 1, 2)$ /3,4/, which admits analytical investigation, at leat with some constraints on the vehicle parameters. Experiments show that monotonically increasing functions $Y_i(\delta_i)$, as well as falling sections of curve $Y_i = Y_i(\delta_i)$ that do not reach the axis of abscissas, are possible in practice. The cubic parabolas shown in the upper part of Fig.5 do not satisfy that condition, but the analysis of the coordinate origin behavior requires information about all singular points, which means that all singularities of curves must be considered. At low velocities $v < v_3$ (the lower part of Fig.5) there are nine singular points including (0.0), when $v > v_3$ there are five such points, since with increasing velocity v points A_5, A_7 and the symmetric to them relative to the coordinate origin points A_6, A_8 vanish. Only points A_1, A_2, A_3, A_4 remain. Points A_1, A_2 lie on the bisectrix of the first and third quadrants of the plane $\delta_1\delta_2$ and, as the coordinate origin, are stationary. Points A_3, A_4 move with change of velocity v, and it is they who play the predominant part in the behavior of the coordinate origin. Their Poincaré indices are -1, while the coordinate origin index was initially equal 1. When $a_1l_1 - a_2l_2 < 0$, the unstable singular points A_3, A_4 move away from the coordinate origin approaching the stationary singular points A_1, A_2 whose indices are 1. At some velocity v_- the indices interchange so that for $v > v_-$ points A_1, A_2 become of the saddle type, and the mobile points A_3, A_4 with the new index $j = 1$ continue to recede from the coordinate origin.

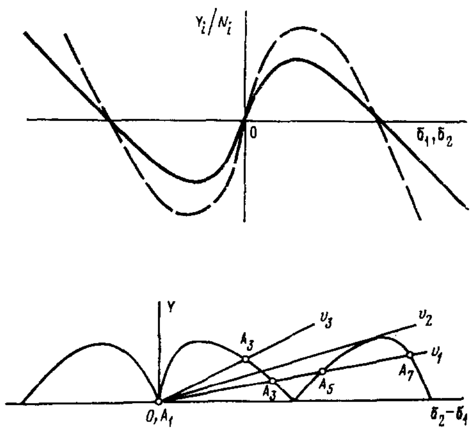

Fig.5

If, however, $a_1l_1 - a_2l_2 > 0$, then for $v < v_+$ the coordinate origin index is unity. As velocity v is increased, the mobile saddle points A_3, A_4 approach the coordinate origin destroying the stability region. When $v = v_+$ singular points merge, as the result of which the coordinate origin becomes at $v > v_+$ equal -1, i.e. it is transformed into an unstable singular point.

The described method enables us not only to analyze the behavior of singular points for any form of dependence $Y_i = Y_i(\delta_i)$ but, also, to control their behavior (suitably selecting that dependence) so as to obtain the beforehand specified properties of the zero solution of equations of perturbed motion.

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Translated by J.J.D.